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Statistical fuzzy approximation by fuzzy positive linear operators

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Abstract

In this paper, we prove a Korovkin-type approximation theorem for fuzzy positive linear operators by using the notion of A-statistical convergence, where A is a non-negative regular summability matrix. This type of approximation enables us to obtain more powerful results than in the classical aspects of approximation theory settings. An application of this result is also given. Furthermore, we compute the rates of this statistical fuzzy convergence of the operators via the fuzzy modulus of continuity.

Keywords: Statistical convergence; Statistical rates; Fuzzy positive linear operators; Fuzzy Korovkin theory; Fuzzy modulus of continuity

1. Introduction

In the classical convergence analysis, almost all terms of a sequence have to belong to an arbitrarily small neighborhood of the limit. The main idea of statistical convergence is to relax this condition and to demand validity of the convergence condition only for a majority of elements. This method of convergence has been investigated in many areas of mathematics. Especially, in recent years, using statistical convergence in the approximation theory has enabled us to obtain more powerful results than the classical aspects (see, for instance, [1–3]). However, as a rule, neither limits nor statistical limits can be calculated or measured with absolute precision. To reflect this imprecision and to model it by mathematical structures, several approaches in mathematics have been developed: fuzzy set theory, fuzzy logic, interval analysis, set valued analysis, etc.

Recently, some approximation theorems in the fuzzy sense have been proved (see [4–9]). The main goal of the present work is to obtain a fuzzy Korovkin theorem by means of statistical convergence and to compute their statistical fuzzy rates. An application of this result is also given. We should remark here that the classical Korovkin theory and its applications may be found in [10–12].

We first collect some basic definitions and results used in the paper.

A fuzzy number is a function $\mu : \mathbb{R} \to [0, 1]$, which is normal, convex, upper semi-continuous and the closure of the set $\mathrm{supp}(\mu)$ is compact, where $\mathrm{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$. The set of all fuzzy numbers are denoted by $\mathbb{R}_{\mathcal{F}}$.

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Let

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}} \quad \text{and} \quad [\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}, \quad (0 < r \leq 1).$$

Then, it is well-known [13] that, for each $r \in [0, 1]$, the set $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For any $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, it is possible to define uniquely the sum $u \oplus v$ and the product $u \odot v$ as follows:

$$[u \oplus v]^r = [u]^r + [v]^r \quad \text{and} \quad [\lambda \odot v]^r = \lambda[u]^r, \quad (0 \leq r \leq 1).$$

Now denote the interval $[u]^r$ by $[u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \leq u_+^{(r)}$ and $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$ for $r \in [0, 1]$. Then, for $u, v \in \mathbb{R}_{\mathcal{F}}$, define

$$u \preceq v \Leftrightarrow u_-^{(r)} \leq v_-^{(r)} \quad \text{and} \quad u_+^{(r)} \leq v_+^{(r)} \quad \text{for all } 0 \leq r \leq 1.$$

Define also the following metric $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ by

$$D(u, v) = \sup_{r \in [0,1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\}.$$

In this case, $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space (see [14]). Let $f, g : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy number valued functions. Then, the distance between f and g is given by

$$D^*(f, g) = \sup_{x \in [a,b]} \sup_{r \in [0,1]} \max \left\{ \left| f_-^{(r)} - g_-^{(r)} \right|, \left| f_+^{(r)} - g_+^{(r)} \right| \right\}.$$

In this article we consider that $j \to \infty$. Nuray and Savaş [15] introduced the fuzzy analog of statistical convergence by using the metric D in the following way. Let $(\mu_n)_{n \in \mathbb{N}}$ be a fuzzy number valued sequence. Then, $(\mu_n)_{n \in \mathbb{N}}$ is called statistically convergent to a fuzzy number μ if, for every $\varepsilon > 0$,

$$\lim_j \frac{\# \{n \leq j : D(\mu_n, \mu) \geq \varepsilon\}}{j} = 0$$

holds, where the symbol $\# \{B\}$ denotes the cardinality of a set B. This limit is denoted by $st - \lim_n D(\mu_n, \mu) = 0$. Recall that the original definition of statistical convergence for number sequences is due to Fast [16] (see also [17, 18]).

Now let $A = (a_{jn})$ be an infinite summability matrix. For a given sequence $x := (x_n)$, the A-transform of x, denoted by $Ax := ((Ax)_j)_{j \in \mathbb{N}}$, is given by

$$(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$$

provided the series converges for each j. We say that A is regular if $\lim Ax = L$ whenever $\lim x = L$ (see, for instance, [19]). Assume that $A = (a_{jn})$ is a non-negative regular summability matrix. Then, the above definition can easily be generalized, the so-called A-statistical convergence, as follows: $(\mu_n)_{n \in \mathbb{N}}$ is A-statistically convergent to $\mu \in \mathbb{R}_{\mathcal{F}}$, which is denoted by $st_A - \lim_n D(\mu_n, \mu) = 0$, if for every $\varepsilon > 0$,

$$\lim_j \sum_{n : D(\mu_n, \mu) \geq \varepsilon} a_{jn} = 0$$

holds. Observe that if we take $A = C_1 = (c_{jn})$, the Cesáro matrix of order one, defined by

$$c_{jn} := c_{jn} = \begin{cases} \dfrac{1}{j}, & \text{if } 1 \leq n \leq j \\ 0, & \text{otherwise.} \end{cases}$$

Then C_1-statistical convergence coincides with the statistical convergence mentioned above. Furthermore, if A is replaced by the identity matrix, then we have the fuzzy convergence introduced by [20]. Some basic results regarding A-statistical convergence for number sequences may be found in the papers [21,22].

2. Statistical fuzzy Korovkin theory

In this section we prove a fuzzy Korovkin-type theorem via the concept of A-statistical convergence. In order to show that our result is stronger than its classical case we display an example of fuzzy positive linear operators by using fuzzy Bernstein polynomials.

Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy number valued functions. Then f is said to be fuzzy continuous at $x_0 \in [a, b]$ provided that whenever $x_n \to x_0$, then $D(f(x_n), f(x_0)) \to \infty$ as $n \to \infty$. Also, we say that f is fuzzy continuous on $[a, b]$ if it is fuzzy continuous at every point $x \in [a, b]$. The set of all fuzzy continuous functions on the interval $[a, b]$ is denoted by $C_{\mathcal{F}}[a, b]$ (see, for instance, [5]). Notice that $C_{\mathcal{F}}[a, b]$ is only a cone, not a vector space. Now let $L : C_{\mathcal{F}}[a, b] \to C_{\mathcal{F}}[a, b]$ be an operator. Then L is said to be fuzzy linear if, for every $\lambda_1, \lambda_2 \in \mathbb{R}$, $f_1, f_2 \in C_{\mathcal{F}}[a, b]$, and $x \in [a, b]$,

$$L(\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2; x) = \lambda_1 \odot L(f_1; x) \oplus \lambda_2 \odot L(f_2; x)$$

holds. Also L is called fuzzy positive linear operator if it is fuzzy linear and, the condition $L(f; x) \preceq L(g; x)$ is satisfied for any $f, g \in C_{\mathcal{F}}[a, b]$ and all $x \in [a, b]$ with $f(x) \preceq g(x)$.

Throughout the paper we use the test functions e_i given by $e_i(x) = x^i$, $i = 0, 1, 2$. Then, the first author [5] obtained the following fuzzy Korovkin theorem.

Theorem A ([5]). *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}[a, b]$ into itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of positive linear operators from $C[a, b]$ into itself with the property*

$$\{L_n(f; x)\}_{\pm}^{(r)} = \tilde{L}_n\left(f_{\pm}^{(r)}; x\right) \tag{2.1}$$

for all $x \in [a, b]$, $r \in [0, 1]$, $n \in \mathbb{N}$ and $f \in C_{\mathcal{F}}[a, b]$. Assume further that

$$\lim_n \left\| \tilde{L}_n(e_i) - e_i \right\| = 0 \quad \textit{for each } i = 0, 1, 2.$$

Then, for all $f \in C_{\mathcal{F}}[a, b]$, we have

$$\lim_n D^*(L_n(f), f) = 0.$$

Then, we get the following result.

Theorem 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}[a, b]$ into itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of positive linear operators from $C[a, b]$ into itself with the property (2.1). Assume further that*

$$st_A - \lim_n \left\| \tilde{L}_n(e_i) - e_i \right\| = 0 \quad \textit{for each } i = 0, 1, 2. \tag{2.2}$$

Then, for all $f \in C_{\mathcal{F}}[a, b]$, we have

$$st_A - \lim_n D^*(L_n(f), f) = 0.$$

Proof. Let $f \in C_{\mathcal{F}}[a, b]$, $x \in [a, b]$ and $r \in [0, 1]$. By the hypothesis, since $f_{\pm}^{(r)} \in C[a, b]$, we may write, for every $\varepsilon > 0$, that there exists a number $\delta > 0$ such that $\left|f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(x)\right| < \varepsilon$ holds for every $y \in [a, b]$ satisfying $|y - x| < \delta$. Then we immediately get, for all $y \in [a, b]$, that

$$\left|f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(x)\right| \le \varepsilon + 2M_{\pm}^{(r)} \frac{(y - x)^2}{\delta^2},$$

where $M_{\pm}^{(r)} := \left\|f_{\pm}^{(r)}\right\|$. Now using the linearity and the positivity of the operators $\tilde{L}_n$, we have, for each $n \in \mathbb{N}$, that

$$\begin{aligned}\left|\tilde{L}_n\left(f_{\pm}^{(r)}; x\right) - f_{\pm}^{(r)}(x)\right| &\le \tilde{L}_n\left(\left|f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(x)\right|; x\right) + M_{\pm}^{(r)}\left|\tilde{L}_n(e_0; x) - e_0(x)\right| \\ &\le \varepsilon + \left(\varepsilon + M_{\pm}^{(r)}\right)\left|\tilde{L}_n(e_0; x) - e_0(x)\right| + \frac{2M_{\pm}^{(r)}}{\delta^2}\left|\tilde{L}_n\left((y - x)^2; x\right)\right|\end{aligned}$$

which yields

$$\left|\tilde{L}_n\left(f_{\pm}^{(r)};x\right)-f_{\pm}^{(r)}(x)\right| \le \varepsilon+\left(\varepsilon+M_{\pm}^{(r)}+\frac{2c^2M_{\pm}^{(r)}}{\delta^2}\right)\left|\tilde{L}_n\left(e_0;x\right)-e_0(x)\right| + \frac{4cM_{\pm}^{(r)}}{\delta^2}\left|\tilde{L}_n\left(e_1;x\right)-e_1(x)\right|+\frac{2M_{\pm}^{(r)}}{\delta^2}\left|\tilde{L}_n\left(e_2;x\right)-e_2(x)\right|,$$

where $c := \max\{|a|,|b|\}$. Also letting $K_{\pm}^{(r)}(\varepsilon) := \max\{\varepsilon+M_{\pm}^{(r)}+\frac{2c^2M_{\pm}^{(r)}}{\delta^2}, \frac{4cM_{\pm}^{(r)}}{\delta^2}, \frac{2M_{\pm}^{(r)}}{\delta^2}\}$ taking supremum over $x\in[a,b]$, the above inequality implies that

$$\left\|\tilde{L}_n\left(f_{\pm}^{(r)}\right)-f_{\pm}^{(r)}\right\| \le \varepsilon+K_{\pm}^{(r)}(\varepsilon)\left\{\left\|\tilde{L}_n\left(e_0\right)-e_0\right\|+\left\|\tilde{L}_n\left(e_1\right)-e_1\right\|+\left\|\tilde{L}_n\left(e_2\right)-e_2\right\|\right\}. \tag{2.3}$$

Now it follows from (2.1) that

$$\begin{aligned} D^*\left(L_n(f),f\right) &= \sup_{x\in[a,b]} D\left(L_n(f;x),f(x)\right) \\ &= \sup_{x\in[a,b]}\sup_{r\in[0,1]}\max\left\{\left|\tilde{L}_n\left(f_-^{(r)};x\right)-f_-^{(r)}(x)\right|,\left|\tilde{L}_n\left(f_+^{(r)};x\right)-f_+^{(r)}(x)\right|\right\} \\ &= \sup_{r\in[0,1]}\max\left\{\left\|\tilde{L}_n\left(f_-^{(r)}\right)-f_-^{(r)}\right\|,\left\|\tilde{L}_n\left(f_+^{(r)}\right)-f_+^{(r)}\right\|\right\}. \end{aligned}$$

Combining the above equality with (2.3), we have

$$D^*\left(L_n(f),f\right) \le \varepsilon+K(\varepsilon)\left\{\left\|\tilde{L}_n\left(e_0\right)-e_0\right\|+\left\|\tilde{L}_n\left(e_1\right)-e_1\right\|+\left\|\tilde{L}_n\left(e_2\right)-e_2\right\|\right\}, \tag{2.4}$$

where $K(\varepsilon) := \sup_{r\in[0,1]}\max\{K_-^{(r)}(\varepsilon), K_+^{(r)}(\varepsilon)\}$. Now, for a given $\varepsilon' > 0$, chose $\varepsilon > 0$ such that $0 < \varepsilon < \varepsilon'$, and also define the following sets:

$$\begin{aligned} U &:= \left\{n\in\mathbb{N} : D^*\left(L_n(f),f\right)\ge\varepsilon'\right\}, \\ U_0 &:= \left\{n\in\mathbb{N} : \left\|\tilde{L}_n\left(e_0\right)-e_0\right\|\ge\frac{\varepsilon'-\varepsilon}{3K(\varepsilon)}\right\}, \\ U_1 &:= \left\{n\in\mathbb{N} : \left\|\tilde{L}_n\left(e_1\right)-e_1\right\|\ge\frac{\varepsilon'-\varepsilon}{3K(\varepsilon)}\right\}, \\ U_2 &:= \left\{n\in\mathbb{N} : \left\|\tilde{L}_n\left(e_2\right)-e_2\right\|\ge\frac{\varepsilon'-\varepsilon}{3K(\varepsilon)}\right\}. \end{aligned}$$

Then inequality (2.4) gives

$$U\subseteq U_0\cup U_1\cup U_2,$$

which guarantees that, for each $j\in\mathbb{N}$,

$$\sum_{n\in U}a_{jn}\le\sum_{n\in U_0}a_{jn}+\sum_{n\in U_1}a_{jn}+\sum_{n\in U_2}a_{jn}. \tag{2.5}$$

If we take limit as $j\to\infty$ on the both sides of inequality (2.5) and use the hypothesis (2.2), we immediately see that

$$\lim_j\sum_{n\in U}a_{jn}=0,$$

whence the result. □

Remark 2.1. If we replace the matrix A in Theorem 2.1 by the identity matrix, then we get Theorem A given by Anastassiou in [5] at once. However, we can construct a sequence of fuzzy positive linear operators, which satisfies our statistical fuzzy approximation result (Theorem 2.1), but not Theorem A.

Take $A = C_1 = (c_{jn})$, the Cesáro matrix of order one and define the sequence (u_n) by:

$$u_n = \begin{cases} 1, & \text{if } n \neq m^2, \ (m = 1, 2, \ldots), \\ \sqrt{n}, & \text{otherwise.} \end{cases} \tag{2.6}$$

Then consider the fuzzy Bernstein-type polynomials as follows:

$$B_n^{(\mathcal{F})}(f; x) = u_n \odot \bigoplus_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \odot f\left(\frac{k}{n}\right),$$

where $f \in C_{\mathcal{F}}[0, 1]$, $x \in [0, 1]$ and $n \in \mathbb{N}$. In this case, we write

$$\left\{B_n^{(\mathcal{F})}(f; x)\right\}_{\pm}^{(r)} = \tilde{B}_n\left(f_{\pm}^{(r)}; x\right) = u_n \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f_{\pm}^{(r)}\left(\frac{k}{n}\right),$$

where $f_{\pm}^{(r)} \in C[0, 1]$. Observe easily that

$$\tilde{B}_n(e_0; x) = u_n,$$
$$\tilde{B}_n(e_1; x) = x u_n,$$
$$\tilde{B}_n(e_2; x) = \left(x^2 + \frac{x(1-x)}{n}\right) u_n.$$

Since

$$\sum_{n:|u_n-1|\geq\varepsilon} c_{jn} = \sum_{n:|u_n-1|\geq\varepsilon} \frac{1}{j} \leq \frac{\sqrt{j}}{j} = \frac{1}{\sqrt{j}} \to 0 \quad (\text{as } j \to \infty),$$

we get

$$st_{C_1} - \lim_n u_n = 1.$$

The above imply that

$$st_{C_1} - \lim_n \left\| \tilde{B}_n(e_i) - e_i \right\| = 0 \quad \text{for each } i = 0, 1, 2.$$

So, by Theorem 2.1, we obtain, for all $f \in C_{\mathcal{F}}[0, 1]$, that

$$st_{C_1} - \lim_n D^*\left(B_n^{(\mathcal{F})}(f), f\right) = 0.$$

However, since the sequence $(u_n)_{n\in\mathbb{N}}$ given by (2.6) is non-convergent (in the usual sense), the sequence $\{B_n^{(\mathcal{F})}(f)\}_{n\in\mathbb{N}}$ is not fuzzy convergent to f.

3. Statistical fuzzy rates

This section is devoted to computing the rates of A-statistical fuzzy convergence in Theorem 2.1. Before starting, we recall that various ways of defining rates of convergence in the A-statistical sense have been introduced in [2] as follows.

Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $(p_n)_{n\in\mathbb{N}}$ be a positive non-increasing sequence of real numbers. Then

(a) A sequence $x = (x_n)$ is A-statistically convergent to the number L with the rate of $o(p_n)$ if for every $\varepsilon > 0$,

$$\lim_j \frac{1}{p_j} \sum_{n:|x_n-L|\geq\varepsilon} a_{jn} = 0.$$

In this case we write $x_n - L = st_A - o(p_n)$ as $n \to \infty$.

(b) If for every $\varepsilon > 0$,

$$\sup_j \frac{1}{p_j} \sum_{n:|x_n|\geq\varepsilon} a_{jn} < \infty,$$

then $(x_n)_{n\in\mathbb{N}}$ is A-statistically bounded with the rate of $O(p_n)$ and it is denoted by $x_n = st_A - O(p_n)$ as $n \to \infty$.

(c) $(x_n)_{n\in\mathbb{N}}$ is A-statistically convergent to L with the rate of $o_m(p_n)$, denoted by $x_n - L = st_A - o_m(p_n)$ as $n \to \infty$, if for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n-L|\geq\varepsilon p_n} a_{jn} = 0.$$

(d) $(x_n)_{n\in\mathbb{N}}$ is A-statistically bounded with the rate of $O_m(p_n)$ provided that there is a positive number M satisfying

$$\lim_j \sum_{n:|x_n|\geq M p_n} a_{jn} = 0,$$

which is denoted by $x_n = st_A - O_m(p_n)$ as $n \to \infty$.

In definitions (a) and (b) the "rate" is more controlled by the entries of the summability method rather than the terms of the sequence $(x_n)_{n\in\mathbb{N}}$. For instance, when one takes the identity matrix I, if we choose any non-increasing sequence $(p_n)_{n\in\mathbb{N}}$ satisfying $1/p_n \leq M$ for some $M > 0$ and for each $n \in \mathbb{N}$, then $x_n - L = st_A - o(p_n)$ as $n \to \infty$ for any convergent sequence $(x_n - L)_{n\in\mathbb{N}}$ regardless of how slowly it goes to zero. To avoid such an unfortunate situation one may borrow the concept of convergence in measure from measure theory to define the rate of convergence as in definitions (c) and (d). So, we use the notations o_m and O_m, respectively.

Notice that, for the convergence of fuzzy number valued sequences or fuzzy number valued function sequences, we have to use the metrics D and D^* instead of the absolute value metric in all definitions mentioned above.

Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$. Then the (first) fuzzy modulus of continuity of f, which is introduced by [9] (see also [5]), is defined by

$$w_1^{(\mathcal{F})}(f, \delta) := \sup_{x,y\in[a,b];\,|x-y|\leq\delta} D\left(f(x), f(y)\right)$$

for any $0 < \delta \leq b - a$.

With this terminology, we have the following result.

Theorem 3.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}[a, b]$ into itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of positive linear operators from $C[a, b]$ into itself with the property (2.1). Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are positive non-increasing sequences and also that the operators $\tilde{L}_n$ satisfy the following conditions:*

(i) $\left\|\tilde{L}_n(e_0) - e_0\right\| = st_A - o(a_n)$ *as* $n \to \infty$,

(ii) $w_1^{(\mathcal{F})}(f, \mu_n) = st_A - o(b_n)$ *as* $n \to \infty$, *where* $\mu_n = \sqrt{\|\tilde{L}_n(\varphi)\|}$ *and* $\varphi(y) = (y - x)^2$ *for each* $x \in [a, b]$.

Then, for all $f \in C_{\mathcal{F}}[a, b]$, *we have*

$$D^*\left(L_n(f), f\right) = st_A - o(c_n) \quad \text{as } n \to \infty,$$

where $c_n := \max\{a_n, b_n\}$ *for each* $n \in \mathbb{N}$. *Furthermore, similar results hold when little "o" is replaced by big "O".*

Proof. By Theorem 3 of [5], one can get, for each $n \in \mathbb{N}$ and $f \in C_{\mathcal{F}}[a, b]$, that

$$D^*\left(L_n(f), f\right) \leq M \left\|\tilde{L}_n\left(e_0\right) - e_0\right\| + \left\|\tilde{L}_n\left(e_0\right) + e_0\right\| w_1^{(\mathcal{F})}(f, \mu_n),$$

where $M := D^*\left(f, \chi_{\{0\}}\right)$ and $\chi_{\{0\}}$ denotes the neutral element for $\oplus$. Then we may write that

$$D^*\left(L_n(f), f\right) \leq M \left\|\tilde{L}_n\left(e_0\right) - e_0\right\| + \left\|\tilde{L}_n\left(e_0\right) - e_0\right\| w_1^{(\mathcal{F})}(f, \mu_n) + 2w_1^{(\mathcal{F})}(f, \mu_n). \tag{3.1}$$

Now, for a given $\varepsilon > 0$, consider the following sets:

$$V := \left\{ n \in \mathbb{N} : D^* \left(L_n(f), f \right) \geq \varepsilon \right\},$$
$$V_0 := \left\{ n \in \mathbb{N} : \left\| \tilde{L}_n \left(e_0 \right) - e_0 \right\| \geq \frac{\varepsilon}{3M} \right\},$$
$$V_1 := \left\{ n \in \mathbb{N} : \left\| \tilde{L}_n \left(e_0 \right) - e_0 \right\| w_1^{(\mathcal{F})}(f, \mu_n) \geq \frac{\varepsilon}{3} \right\},$$
$$V_2 := \left\{ n \in \mathbb{N} : w_1^{(\mathcal{F})}(f, \mu_n) \geq \frac{\varepsilon}{6} \right\}.$$

Then, by (3.1), observe that $V \subseteq V_0 \cup V_1 \cup V_2$. Also defining

$$V_1' := \left\{ n \in \mathbb{N} : \left\| \tilde{L}_n \left(e_0 \right) - e_0 \right\| \geq \sqrt{\frac{\varepsilon}{3}} \right\},$$
$$V_1'' := \left\{ n \in \mathbb{N} : w_1^{(\mathcal{F})}(f, \mu_n) \geq \sqrt{\frac{\varepsilon}{3}} \right\},$$

we immediately see that $V_1 \subseteq V_1' \cup V_1''$, which gives $V \subseteq V_0 \cup V_1' \cup V_1'' \cup V_2$. Then we get

$$\frac{1}{c_j} \sum_{n \in V} a_{jn} \leq \frac{1}{c_j} \sum_{n \in V_0} a_{jn} + \frac{1}{c_j} \sum_{n \in V_1'} a_{jn} + \frac{1}{c_j} \sum_{n \in V_1''} a_{jn} + \frac{1}{c_j} \sum_{n \in V_2} a_{jn}. \tag{3.2}$$

Since $c_j = \max\{a_j, b_j\}$, we obtain from (3.2) that

$$\frac{1}{c_n} \sum_{n \in V} a_{jn} \leq \frac{1}{a_j} \sum_{n \in V_0} a_{jn} + \frac{1}{a_j} \sum_{n \in V_1'} a_{jn} + \frac{1}{b_j} \sum_{n \in V_1''} a_{jn} + \frac{1}{b_j} \sum_{n \in V_2} a_{jn}. \tag{3.3}$$

So, taking the limit as $j \to \infty$ in (3.3) and using the hypotheses (i) and (ii), we get

$$\lim_j \frac{1}{c_j} \sum_{n \in V} a_{jn} = 0,$$

which completes the proof. □

Similarly as in the proof of Theorem 3.1, one can get the following result at once.

Theorem 3.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n \in \mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}[a, b]$ into itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n \in \mathbb{N}}$ of positive linear operators from $C[a, b]$ into itself with the property (2.1). Suppose that $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are positive non-increasing sequences and also that the operators $\tilde{L}_n$ satisfy the following conditions:*

(i) $\left\| \tilde{L}_n(e_0) - e_0 \right\| = st_A - o_m(a_n)$ *as* $n \to \infty$,

(ii) $w_1^{(\mathcal{F})}(f, \mu_n) = st_A - o_m(b_n)$ *as* $n \to \infty$.

Then, for all $f \in C_{\mathcal{F}}[a, b]$, we have

$$D^* \left(L_n(f), f \right) = st_A - o(d_n) \quad \text{as } n \to \infty,$$

where $d_n := \max\{a_n, b_n, a_n b_n\}$ for each $n \in \mathbb{N}$. Furthermore, similar results hold when little "o_m" is replaced by big "O_m".

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